Calogero-Sutherland model from excitations of Chern-Simons vortices

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Abstract

We consider a large-N Chern-Simons theory for the attractive bosonic matter (Jackiw-Pi model) in the Hamiltonian, collective-field approach based on the 1/N expansion. We show that the dynamics of density excitations around the ground-state semiclassical configuration is governed by the Calogero or by the Sutherland Hamiltonian, depending on the symmetry of the underlying static-soliton configuration. The relationship between the Chern-Simons coupling constant λ and the Calogero-Sutherland statistical parameter λ_c signalizes some sort of statistical transmutation accompanying the dimensional reduction of the initial problem.

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Gauge models of a scalar field with the Chern-Simons term [1] in $2+1$ space-time dimensions are known to support soliton or vortex solutions [2,3]. By using the nonrelativistic field theory of the self-attracted bosonic matter minimally coupled to an Abelian Chern-Simons gauge field, the authors of Ref. [2] have shown that there exists a static self-dual soliton solution for a specific choice of the coupling constant. We have rederived this soliton solution in the collective-field approach by including higher-order terms in the $1/N$ expansion [4]. In our approach, this soliton saturates the Bogomol'nyi bound and does not receive quantum corrections to its energy in the next-to-leading approximation. Furthermore, owing to the fact that our soliton is normalized to a large number N, being the number of bosonic particles, we can describe it as a spiky distribution effectively representing one-dimensional bubbles living on the xy plane. This observation substantially simplifies the problem of quantum excitations about such a configuration and allows us to identify their dynamics with that of the Calogero-Sutherland model.

There exist several recent papers that elucidate the connection between the Chern-Simons-based anyonic physics in the fractional quantum Hall effect (FQHE) and the Calogero-Sutherland model.

It was noted in [5] that there was a close similarity between the Calogero model [6] and the system of anyons at the lowest Landau level in a strong external magnetic field. It was conjectured in Ref. [7] and later proved in Ref. [8] that the two systems were in fact equivalent. The equivalence was demonstrated on the algebraic grounds, i.e., by finding a complex, Bargmann-Fock representation of the underlying operator algebras.

Furthermore, it was shown that the ground state of the Calogero-Sutherland (CS) [9] model and the Laughlin state for the FQHE coincided exactly in the narrow-cylinder geometry [10,11]. It was also shown that there was a similarity between the edge states of a non-narrow droplet of the FQHE and the chiral Tomonaga-Luttinger liquid whose exponent was equal to that of the chiral-constrained CS model [12,13].

Using the hydrodynamic collective-field theory, the authors of the Ref. [14] were able to show that the fermion correlation functions along the boundaries of the FQH droplet

were interpolated to the correlation functions of the CS model as the droplet width was continuously narrowed. Finally, the FQH effect and the CS model shared the same, infinite-dimensional W_∞ algebra [15–17].

In this paper we would like to extend this equivalence to a completely different physical situation. Using the collective-field theory approach, we show that the Jackiw-Pi model which describes nonrelativistic anyons interacting via the δ-function attractive potential undergoes a dinamical reduction in dimensionality, i.e., it reduces to a one-dimensional CS system.

The Hamiltonian for N spinless bosonic particles in the presence of the vortex of the strength v, located at the point Z [4], is

$$\begin{aligned} H = & -2\sum_{i=1}^{N}\frac{\partial^2}{\partial z_i\partial \bar{z}_i} + \frac{\lambda^2}{2}\sum_{i=1}^{N}\left|\sum_{j\neq i}^{N}\frac{1}{z_i-z_j}\right|^2 - \lambda\sum_{i\neq j}^{N}\frac{1}{z_i-z_j}\frac{\partial}{\partial \bar{z}_i} \\ & +\lambda\sum_{i\neq j}^{N}\frac{1}{\bar{z}_i-\bar{z}_j}\frac{\partial}{\partial z_i} - 2v\sum_{i=1}^{N}\frac{1}{\bar{z}_i-\bar{Z}}\frac{\partial}{\partial z_i} - v\lambda\sum_{i\neq j}^{N}\frac{1}{z_i-z_j}\frac{1}{\bar{z}_i-\bar{Z}} + V, \end{aligned} \tag{1}$$

where the potential V depends only on the position of particles. The complex numbers $z_i = x_i + iy_i$ represent the position of the i-th particle. ¿From the bosonic wave function we have extracted the prefactor given by $\prod_{i=1}^{N}(\bar{z}_i - \bar{Z})^v$, which represents the afore-mentioned vortex. We have already shown in [4] that, in the leading order in N, the collective motion of the system is given by the classical solution of collective-field theory. For the sake of clarity, we review this part of Ref. [4] once again. The collective-field approach to the Jackiw-Pi anyonic system is described by the Hamiltonian

$$\begin{aligned} H = & \frac{1}{2}\int d^2\mathbf{r}\rho(\mathbf{r})\left\{\nabla\pi(\mathbf{r}) + \hat{n}\times\left[\frac{1}{2}\frac{\nabla\rho(\mathbf{r})}{\rho(\mathbf{r})} + \lambda\int d^2\mathbf{r}'\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2} - v\frac{\mathbf{r}-\mathbf{R}}{|\mathbf{r}-\mathbf{R}|^2}\right]\right\}^2 \\ & +\lambda\pi\int d^2\mathbf{r}\rho^2(\mathbf{r}) + V, \end{aligned} \tag{2}$$

where the dimensionless constant λ is the so-called statistical parameter which is to tune the desired statistics, and $\hat{n}$ is the unit vector perpendicular to the plane in which particles move. The collective field $\rho(\mathbf{r})$ is the continuum limit of the dynamical quantity:

$$\rho(\mathbf{r}) = \sum_{i=1}^{N}\delta(\mathbf{r}-\mathbf{r}_i), \tag{3}$$

where $\mathbf{r}_i$ are the positions of N bosonic particles interacting through the long-range statistical Bohm-Aharonov-like vector potential

$$\mathbf{A}(\mathbf{r}) = \lambda \hat{n} \times \int d^2\mathbf{r}' \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2}. \tag{4}$$

The operator $\pi(\mathbf{r})$ is the canonical conjugate of the field $\rho(\mathbf{r})$:

$$[\nabla \pi(\mathbf{r}), \rho(\mathbf{r}')] = -i \nabla \delta(\mathbf{r} - \mathbf{r}'). \tag{5}$$

The v-dependent term reflects the vortex-type singularity which should be canceled by the $\nabla \ln \rho$ term at the point of the vanishig density $\rho(\mathbf{r})$, i.e., at $\mathbf{r} = \mathbf{R}$.

If we had extracted the prefactor $\prod_{i=1}^{N} (z_i - Z)^v$ from the particle wave function, we would have obtained the same Hamiltonian , the only difference being in the sign of the λ-dependent terms. It will later become apparent that this form of the effective Hamiltonian describes vortices with the negative statistical parameter λ.

If we fine-tune the coupling g of the δ-function potential V

$$V = -g \sum_{i,j}^{N} \delta(z_i - z_j) \tag{6}$$

and choose $g = |\lambda|$, then the Hamiltonian (2) becomes

$$H = \frac{1}{2} \int d^2\mathbf{r} \rho(\mathbf{r}) \left\{ \nabla \pi(\mathbf{r}) + \hat{n} \times \left[\frac{1}{2} \frac{\nabla \rho(\mathbf{r})}{\rho(\mathbf{r})} + |\lambda| \int d^2\mathbf{r}' \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} - v \frac{\mathbf{r} - \mathbf{R}}{|\mathbf{r} - \mathbf{R}|^2} \right] \right\}^2 . \tag{7}$$

The leading part of the collective-field Hamiltonian in the $1/N$ expansion is given by the effective potential

$$V_{\text{eff}} = \frac{1}{2} \int d^2\mathbf{r} \rho(\mathbf{r}) \left[\frac{1}{2} \frac{\nabla \rho(\mathbf{r})}{\rho(\mathbf{r})} + |\lambda| \int d^2\mathbf{r}' \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} - v \frac{\mathbf{r} - \mathbf{R}}{|\mathbf{r} - \mathbf{R}|^2} \right]^2 . \tag{8}$$

Owing to the positive definiteness of the effective potential (8), the Bogomol'nyi limit appears. The Bogomol'nyi bound is saturated by the positive normalizable solution $\rho_0(\mathbf{r})$ of the equation

$$\frac{1}{2} \frac{\nabla \rho_0(\mathbf{r})}{\rho_0(\mathbf{r})} + |\lambda| \int d^2\mathbf{r}' \rho_0(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} - v \frac{\mathbf{r} - \mathbf{R}}{|\mathbf{r} - \mathbf{R}|^2} = 0. \tag{9}$$

Let us now in more detail examin the static solutions of the Bogomol'nyi equation (9) and the corresponding excitations whose dynamics will be shown to be equal to that of the CS model. In the rectangular coordinates (x, y), Eq. (9) can be written in the form

$$\frac{1}{2}\frac{\partial}{\partial x}\ln\rho_0(\mathbf{r}) + |\lambda|\int d^2\mathbf{r}'\rho_0(\mathbf{r}')\frac{x-x'}{|\mathbf{r}-\mathbf{r}'|^2} - v\frac{x-X}{|\mathbf{r}-\mathbf{R}|^2} = 0, \tag{10a}$$

$$\frac{1}{2}\frac{\partial}{\partial y}\ln\rho_0(\mathbf{r}) + |\lambda|\int d^2\mathbf{r}'\rho_0(\mathbf{r}')\frac{y-y'}{|\mathbf{r}-\mathbf{r}'|^2} - v\frac{y-Y}{|\mathbf{r}-\mathbf{R}|^2} = 0. \tag{10b}$$

There exists an interesting solution to this coupled set of equations, depending only on one variable, let us say x, for definiteness. Since the integral kernel in the second equation is an odd function in $y - y'$, the set (10) is consistent only for $|\mathbf{R}| \to \infty$. The only relevant equation is therefore given by

$$\frac{1}{2}\frac{\partial}{\partial x}\ln\rho_0(x) + |\lambda|\pi\int dx'\rho_0(x')\mathrm{sign}(x-x') = 0, \tag{11}$$

where we have used the result valid for infinite space:

$$\int \frac{dy'}{(x-x')^2+(y-y')^2} = \frac{\pi}{|x-x'|}, \quad x \neq x'. \tag{12}$$

The integro-differential equation (11) can be reduced to a differential one by taking the derivative with respect to x:

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\ln\rho_0(x) + 2|\lambda|\pi\rho_0(x) = 0. \tag{13}$$

This equation has a positive and normalizable solution given by

$$\rho_0(x) = \frac{N\kappa}{\cosh^2 2\kappa x}, \quad \kappa = \frac{|\lambda|\pi N}{2}. \tag{14}$$

It is interesting to note that our soliton solution (14) can be obtained as a special case of the general solution to the Liouville equation. In fact, applying the gradient operator, we can transform Eq. (9) into the Liouville equation:

$$\frac{1}{2}\Delta\ln\rho_0(\mathbf{r}) + 2|\lambda|\pi\rho_0(\mathbf{r}) = 0. \tag{15}$$

It is known that the general, positive solution to this equation is given by

$$\rho_0(\mathbf{r}) = \frac{2}{|\lambda|\pi} \frac{\left|\frac{df(z)}{dz}\right|^2}{[1+|f(z)|^2]^2}, \tag{16}$$

where $f(z)$ is an arbitrary holomorphic function of $z = x + iy$, but chosen so that $\rho_0(\mathbf{r})$ is nonsingular and nonvanishing except at infinity. It is easy to see that the only choice for $f(z)$ which generates a single-variable, x-dependent solution is given by

$$f(z) = \mathrm{e}^{az}, \tag{17}$$

where a is an arbitrary real constant. The requirement of normalizability fixes the constant to be $a = 2\kappa$. This finally reproduces our solution (14).

The collective-field configuration (14) describes the ground state of N bosonic particles with the attractive $\delta-$function interaction, as can be easily seen from the corresponding one-dimensional effective potential

$$V_{\text{eff}} = \frac{1}{2}\int dx \rho(x) \left[\frac{1}{2}\frac{\partial}{\partial x}\ln\rho(x) + |\lambda|\pi \int dx' \rho(x') \mathrm{sign}(x-x')\right]^2. \tag{18}$$

Actually, the δ interaction appears as the cross term in the square (18):

$$\begin{aligned} V_{\text{eff}} &= \frac{1}{8}\int dx \frac{1}{\rho(x)}\left(\frac{\partial\rho(x)}{\partial x}\right)^2 - |\lambda|\pi \int dx \rho^2(x) \\ &+ \frac{\lambda^2\pi^2}{2}\int dx\rho(x)\left[\int dx'\rho(x')\mathrm{sign}(x-x')\right]^2. \end{aligned} \tag{19}$$

Using the identity

$$\mathrm{sign}(x-y)\mathrm{sign}(x-z) + \mathrm{sign}(y-x)\mathrm{sign}(y-z) + \mathrm{sign}(z-x)\mathrm{sign}(z-y) = 1, \tag{20}$$

one can show that the contribution of the last term in (19) transforms into an irrelevant constant which only shifts the zero point of the energy scale. With increasing number of particles N, the soliton profile of width proportional to $1/N$ (14) becomes thinner, finally taking the form of the δ distribution:

$$\rho_0(x) = N\delta(x). \tag{21}$$

This can be readily obtained by using one of the appropriate representations of the δ function:

$$\delta(x) = \lim_{\epsilon \to 0} \frac{\exp(x/\epsilon)}{\epsilon[1 + \exp(x/\epsilon)]^2} \ , \ \epsilon = \frac{1}{4\kappa}. \tag{22}$$

Consequently, particles are restricted by their statistical interaction (effectively, the attractive δ-function interaction) to move along the y axis. Although the motion of particles takes place in a two-dimensional space, the system is effectively one-dimensional. The only relevant degree of freedom we are left with can be described by the residual collective-field excitation $\tilde{\rho}(y)$, which also lives on the y axis, i.e.,

$$\rho(\mathbf{r}) = \delta(x)\tilde{\rho}(y). \tag{23}$$

Having written the excited collective-field configuration in the form (23), we automatically normalize the residual field $\tilde{\rho}$ to the same number of particles N.

To find the dynamics of this excitation, we must insert the factorization form (23) into the collective Hamiltonian (2). A simple calculation yields

$$H = \frac{1}{2} \int dx dy \rho_0(x)\tilde{\rho}(y) \left[\frac{\partial \pi}{\partial x} - \frac{1}{2}\frac{\partial}{\partial y} \ln \tilde{\rho}(y) - |\lambda| \int dx' dy' \rho_0(x')\tilde{\rho}(y') \frac{y - y'}{(x - x')^2 + (y - y')^2} \right]^2$$
$$+ \frac{1}{2} \int dx dy \rho_0(x)\tilde{\rho}(y) \left[\frac{\partial \pi}{\partial y} + \frac{1}{2}\frac{\partial}{\partial x} \ln \rho_0(x) + |\lambda| \int dx' dy' \rho_0(x')\tilde{\rho}(y') \frac{x - x'}{(x - x')^2 + (y - y')^2} \right]^2 . \tag{24}$$

If we take into account the soliton equation (13) and the limiting form of the corresponding solution (21), we can show that the collective Hamiltonian is

$$H = \frac{1}{2} \int dy \tilde{\rho}(y) \left(\frac{\partial \pi}{\partial y} \right)^2 + \frac{1}{2} \int dy \tilde{\rho}(y) \left[\frac{1}{2}\frac{\partial}{\partial y} \ln \tilde{\rho}(y) + |\lambda| \int dy' \frac{\tilde{\rho}(y')}{y - y'} \right]^2 . \tag{25}$$

Here we have neglected the x-dependence of the conjugate momentum π since all particles are allowed to move only along the y axis. By rescaling the field $\tilde{\rho}(y) \to c\rho(y)$ and the momentum $\pi(y) \to \pi(y)/c$, we can recast the collective Hamiltonian (25) into the Calogero form [18,19] as

$$H = \frac{1}{c} \left\{ \frac{1}{2} \int dy \rho(y) \left(\frac{\partial \pi}{\partial y} \right)^2 + \frac{1}{2} \int dy \rho(y) \left[\frac{\lambda_c - 1}{2} \frac{\partial}{\partial y} \ln \rho(y) + \lambda_c \int dy' \frac{\rho(y')}{y - y'} \right]^2 \right\} , \tag{26}$$

where the constant c, the anyonic parameter λ and the Calogero statistical parameter λ_c are interrelated by

$$c = \lambda_c - 1 \text{ and } |\lambda| c^2 = \lambda_c, \tag{27}$$

finally leading to the relation

$$\lambda_c = |\lambda|(\lambda_c - 1)^2. \tag{28}$$

It is interesting to observe that for a fixed value of the anyonic statistical parameter λ there are, in principle, two different values of the corresponding Calogero-Sutherland statistical parameter λ_c^+ and λ_c^- connected by the relation $\lambda_c^+ \lambda_c^- = 1$. This relation somehow reflects the duality of the λ_c and $1/\lambda_c$ Calogero-Sutherland models.

Now we are going to show that our system of Jackiw-Pi anyons can be similarly reduced to the Sutherland model. In this case, we are looking for a radially symmetric soliton solution to Eq. (9), describing the vortex located at the origin. It has been shown in [4] that there exists a radially symmetric, positive and normalizable collective-field configuration which minimizes the energy (2). It is given by the vortex form

$$\rho_0(r) = \frac{|\lambda| N^2}{2\pi r^2} \left[\left(\frac{r_0}{r} \right)^{\frac{N|\lambda|}{2}} + \left(\frac{r}{r_0} \right)^{\frac{N|\lambda|}{2}} \right]^{-2}. \tag{29}$$

The vorticity v is fixed by the normalization condition and is given by

$$v = N \frac{|\lambda|}{2} - 1. \tag{30}$$

The parameter r_0 reflects the scale invariance of the problem and cannot be determined. Now, if N is large enough, we can again replace the soliton configuration $\rho_0(r)$ with the δ profile:

$$\rho_0(r) = \frac{N}{2\pi} \frac{\delta(r - r_0)}{r_0}. \tag{31}$$

The residual collective-field excitations $\tilde{\rho}(\varphi)$ can move only along the circle of radius r_0. The corresponding collective Hamiltonian can be found along similar lines, explicitly given for the rectangular geometry:

$$H = \frac{1}{2}\int rdrd\varphi\rho_0(r)\tilde{\rho}(\varphi)\left[\frac{\partial\pi}{\partial r} - \frac{1}{2r}\frac{\partial}{\partial\varphi}\ln\tilde{\rho}(\varphi) - |\lambda|\int r'dr'd\varphi'\rho_0(r')\tilde{\rho}(\varphi')\frac{r'\sin(\varphi-\varphi')}{|\mathbf{r}-\mathbf{r}'|^2}\right]^2$$

$$+\frac{1}{2}\int rdrd\varphi\rho_0(r)\tilde{\rho}(\varphi)\left[\frac{1}{r}\frac{\partial\pi}{\partial\varphi} + \frac{1}{2}\frac{\partial}{\partial r}\ln\rho_0(r) - |\lambda|\int r'dr'd\varphi'\rho_0(r')\tilde{\rho}(\varphi')\frac{r'\cos(\varphi-\varphi')-r}{|\mathbf{r}-\mathbf{r}'|^2} - \frac{v}{r}\right]^2. \quad (32)$$

Using the radial part of the Bogomol'nyi equation (9),

$$\frac{1}{2}\frac{\partial}{\partial r}\ln\rho_0(r) - |\lambda|\int r'dr'd\varphi'\rho_0(r')\frac{r'\cos(\varphi-\varphi')-r}{|\mathbf{r}-\mathbf{r}'|^2} - \frac{v}{r} = 0, \quad (33)$$

and the limiting form of the corresponding solution (31), it can be shown that the collective Hamiltonian for two-dimensional anyons finally reduces to the collective Sutherland Hamiltonian [20]

$$H = \frac{1}{2r_0^2}\int d\varphi\tilde{\rho}(\varphi)\left(\frac{\partial\pi}{\partial\varphi}\right)^2 + \frac{1}{2r_0^2}\int d\varphi\tilde{\rho}(\varphi)\left[\frac{1}{2}\frac{\partial}{\partial\varphi}\ln\tilde{\rho}(\varphi) + \frac{|\lambda|}{2}\int d\varphi\tilde{\rho}(\varphi)\cot\frac{\varphi-\varphi'}{2}\right]^2, \quad (34)$$

up to the irrelevant constant term. Further rescaling finally connects the parameter λ with the Calogero-Sutherland statistical parameter λ_c by the same set of relations (27).

It is evident from the relations (27) that, for example, as anyonic statistics in two dimensions approaches "super" bosonic statistics ($|\lambda| \to 2$), the corresponding statistics of the generated Calogero model goes to bosonic statistics ($\lambda_c \to 2$) or semionic one ($\lambda_c \to 1/2$). Moreover, for $|\lambda| \to \infty$, the statistical parameter of the Calogero model approaches unity and we recover the collective-field Hamiltonian for the $c = 1$ matrix model. This is the signal of some sort of statistical transmutation accompanying the dimensional reduction of the anyonic system. However, the critical value $|\lambda| = 0$ is forbidden because, in this case, there would be no static-soliton solution to Eq. (13), which represents the cornerstone of dimensional reduction.

Further study is still needed to fully understand the physical meaning of this dimensional reduction and the statistical transmutation associated with it.

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REFERENCES

[1] R. Jackiw and S. Templeton, *Phys. Rev.* **D23**, 2291 (1981); J. Schonfeld, *Nucl. Phys.* **B185**, 157 (1981); S. Deser, R. Jackiw, and S. Templeton, *Phys. Rev. Lett.* **48**, 975 (1982).

[2] R. Jackiw and S. Y. Pi, *Phys. Rev. Lett.* **64**, 2969 (1990); *Phys. Rev.* **D42**, 3500 (1990); *Prog. Theor. Phys. Suppl.* 107, 1 (1992).

[3] J. Hong, Y. Kim, and P. Y. Pack, *Phys. Rev. Lett.* **64**, 2230 (1990); R. Jackiw and S. Weinberg, *Phys. Rev. Lett.* **64**, 2234 (1990); Z. F. Ezawa, M. Hotta, and A. Iwazaki, *Phys. Rev.* **D44**, 452 (1991); P. K. Ghosh, *Phys. Rev.* **D49**, 5458 (1994); I. V. Barashenkov and A. O. Harin, *Phys. Rev.* **D52**, 2471 (1995).

[4] I. Andrić and V. Bardek, *Phys. Rev.* **D47**, 5546 (1993).

[5] A. P. Polychronakos, *Phys. Lett.* **B264**, 362 (1991).

[6] F. Calogero, *J. Math. Phys.* **10**, 2191,2197 (1969); *J. Math. Phys.* **12**, 419 (1971).

[7] T. H. Hansson, J. M. Leinaas, and J. Myrheim, *Nucl. Phys.* **B384**, 559 (1992).

[8] L. Brink, T. H. Hansson, S. Konstein, and M. A. Vasiliev, *Nucl. Phys.* **B401**, 591 (1993).

[9] B. Sutherland, *Phys. Rev.* **A4**, 2019 (1971); *Phys. Rev.* **A5**, 1372 (1972); *Phys. Rev. Lett.* **34**, 1083 (1975).

[10] S. Iso, *Mod. Phys. Lett.* **A9**, 2123 (1994).

[11] H. Azuma and S. Iso, *Phys. Lett.* **B331**, 107 (1994).

[12] N. Kawakami, *Phys. Rev. Lett.* **71**, 275 (1993).

[13] X. G. Wen, *Phys. Rev. Lett.* **64**, 2206 (1990).

[14] S. Iso and S. J. Rey, *Phys. Lett.* **B352**, 111 (1995).

[15] S. Iso, D. Karabali, and B. Sakita, *Phys. Lett.* **B296**, 143 (1992).

[16] A. Cappeli, C. A. Trugenberger, and G. R. Zemba, *Nucl. Phys.* **B396**, 465 (1993); *Phys. Rev. Lett.* **72**, 1902 (1994); J. Avan and A. Jevicki, *Phys. Lett.* **B266**, 35 (1991); *Phys. Lett.* **B272**, 17 (1991).

[17] R. Caracciolo, A. Lerda, and G. R. Zemba, *Phys. Lett.* **B352**, 304 (1995).

[18] I. Andrić, A. Jevicki, and H. Levine, *Nucl. Phys.* **B215**, 307 (1983).

[19] I. Andrić and V. Bardek, *J. Phys.* **A21**, 2847 (1988).

[20] I. Andrić, V. Bardek, and L. Jonke, Report No. RBI-TH-2/96, 1996 (unpublished).